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An Asymptotic Theory of Guided Waves (II)

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SUMMARY

A method of obtaining high frequency asymptotic expansions for time harmonic waves propagating in non uniform waveguides is presented. Both ordinary asymptotic expansions, which are not valid near turning points, and uniform expansions which are valid across turning points are presented for various types of boundary conditions. The correspondance between these expansions and the asymptotic expansion of the exact solution of a "canonical problem" is shown. A numerical example demonstrating the usefulness of this method is presented. This paper is a sequel to reference [1].

1. Introduction

In reference [1] which appeared in this journal, we presented a method for finding high frequency asymptotic expansions for time harmonic waves in various guiding structures. This paper is a sequel to [1], improving several of its results. The method consists of assuming a suitable "Ansatz" for the solution of Helmholtz's equation in certain "guiding" domains, subject to linear homogeneous boundary conditions at the boundaries of that domain.

In Section 2 eqs. (2.1)–(2.3) the mathematical problem is stated. The requirement (2.3) is essential from both theoretical and practical considerations. From a practical point of view, if a waveguide is oversized, i.e. the crosssection dimensions are large compared to the wavelength λ, where

$$\lambda = 2\pi/K , \tag{1.1}$$

many modes will propagate. In that case the mode approach to the solution of the field is inferior to the ray approach [2]–[4]. The mode approach becomes preferable when the number of propagating modes is small, which is when the crosssection dimensions are of the order of wavelength. This relation is expressed by (2.3). From a theoretical point of view, the expansion in inverse powers of K which we assume in (2.8), (2.9 a, b) becomes meaningless if the expansion coefficients A_i and B_i ($i=0, 1, 2, \ldots$) are functions of some power of K. It is not hard to show that A_i and B_i will be $O(K^0)$ if (2.3) is satisfied. After statement of the problem (which is more general than in [1]) the formal solution is obtained. Special cases of the general boundary conditions (2.2) are the Dirichlet condition ($Z_j \to \infty$) and the Neumann condition ($Z_j \to 0$). In spite of being simpler, they pose a certain difficulty. This problem is discussed and solved in Section 3.

The asymptotic solutions of [1] and of Sections 2 and 3 are non uniform or "ordinary", since they break down in the vicinity of turning points of the equations (which are the cut off points for some propagating mode). Following [5, 6], a uniform asymptotic expansion that is valid across turning points (or cut off points) is introduced in Section 4.

It is generally hard to show that an assumed formal asymptotic series is indeed the asymptotic expansion of the solution of a given boundary value problem for partial differential equations. Only in very few special cases this has been done [6]. The usual way to gain confidence in formal asymptotic expansions is to apply them to the solution of "canonical problems" whose

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exact solution is known, and compare the asymptotic solution with the asymptotic expansion of the known exact solution. We have done that in Section 5. Finally, to show the usefulness of our method, we have calculated the first term of the asymptotic solution (both "ordinary" and "uniform") of an example which cannot be solved exactly in any known way. The results are given in Section 6.

In this paper we deal with two dimensional problems only. This was done in order to facilitate the somewhat cumbersome calculations. Generalization of the results to three dimensions is not difficult, as was shown in [1].

2. Ordinary Asymptotic Expansion For Impedance Boundary Conditions

We look for solutions to the following problem:

$$[\nabla^2 + K^2 n^2(x)]\, U(X) = 0 \quad X \in R^1\,, \; n(x) \in C^1\,, \qquad (X = (x, Y))\,, \tag{2.1}$$

where R^1 is the region bounded by two given functions $Y = H_1(x)$ and $Y = H_2(x)$ (See Fig. 1 on page 104). The boundary conditions are

$$\frac{\partial U}{\partial v} + iKZ_j(x)U = 0 \text{ on } Y = H_j(x)\,, \qquad j = 1, 2\,. \tag{2.2}$$

$$\nabla^2 = \partial^2/\partial Y^2 + \partial^2/\partial x^2\,,$$

and $\partial/\partial v$ stands for differentiation in the direction of the normal to H_j.

In order that R^1 be a "waveguide", H_1 and H_2 must not intersect, i.e. $H_1 > H_2$. In addition we require that

$$\sup_x (H_1 - H_2) = O(K^{-1})\,. \tag{2.3}$$

Actually we assume that $H_1 - H_2 = K^{-1}F(x)$. If $H_1 - H_2 = G(x; K^{-1})$ we expand G in inverse powers of K and the problem is solved in the same way. $Z_j(x)$ are given functions, sometimes called "surface impedance".

Eq. (2.2) can be rewritten as

$$\left[U_y - \frac{\partial H_j}{\partial x} \cdot \frac{\partial U}{\partial x}\right] + iKZ_j U D_j = 0 \text{ on } Y = H_j(x)\,, \qquad j = 1, 2\,, \tag{2.4}$$

where

$$D_j = D_j(x) = \left\{1 + \left(\frac{\partial H_j}{\partial x}\right)^2\right\}^{\frac{1}{2}}.$$

We now introduce the following change of variables:

$$KY = y\;;\; KH_j = h_j\;;\; U(x, Y) = u(x, y)\,, \quad R^1 \to R\,. \tag{2.5}$$

Under this change of variables our problem becomes

$$L[u] = [\partial^2/\partial x^2 + K^2 \partial^2/\partial y^2 + K^2 n^2(x)]\,u = 0\,, \qquad \forall (x, y) \in R \tag{2.6}$$

$$K^2\left[\frac{\partial u}{\partial y} + iZ_j D_j u\right] - \frac{dh_j}{dx}\frac{\partial u}{\partial x} = 0 \text{ on } y = h_j(x)\,, \qquad j = 1, 2\,. \tag{2.7}$$

We may assume now without loss of generality that the solution u of the boundary value problem (2.6) and (2.7) has the form

$$u = u_1 + u_2 = \{A \sin[q(x)\cdot(y - h_1(x))] + B \sin[q(x)\cdot(y - h_2(x))]\} \cdot e^{iK\sigma(x)}\,. \tag{2.8}$$

We assume that each part of the sum (2.8) satisfies eq. (2.6). We denote for brevity: $\phi_j = q(x)(y - h_j(x))$, $j = 1, 2$, and assume that $A(x, y)$, $B(x, y)$, in (2.8) have the following asymptotic expansion:

$$A(x,y) \sim A_0(x) + \sum_{m=1}^{\infty} A_m(x,y)(iK)^{-m} \tag{2.9a}$$

$$B(x,y) \sim B_0(x) + \sum_{m=1}^{\infty} B_m(x,y)(iK)^{-m} \tag{2.9b}$$

with $A_l, B_l \in C^2 \quad l = 0, 1, 2, \ldots$.

Evaluating the necessary derivatives appearing in (2.6), (2.7) we get:

$$u_x \equiv \frac{\partial u}{\partial x} = [(A \sin\phi_1 + B \sin\phi_2)_x + iK\sigma_x(A\sin\phi_1 + B\sin\phi_2)]\, e^{iK\sigma(x)}$$

$$u_{xx} \equiv \frac{\partial^2 u}{\partial x^2} = \{-K^2(\sigma_x)^2[A\sin\phi_1 + B\sin\phi_2] + iK[2\sigma_x(A\sin\phi_1 + B\sin\phi_2)_x + \\ + \sigma_{xx}(A\sin\phi_1 + B\sin\phi_2)_x] + K^0(A\sin\phi_1 + B\sin\phi_2)_{xx}\}\, e^{iK(x)} .$$

$$u_y \equiv \frac{\partial u}{\partial y} = [A_y \sin\phi_1 + B_y \sin\phi_2 + Aq(x)\cos\phi_1 + Bq(x)\cos\phi_2]\, e^{iK\sigma(x)}$$

$$u_{yy} \equiv \frac{\partial^2 u}{\partial y^2} = [A_{yy}\sin\phi_1 + B_{yy}\sin\phi_2 + 2A_y q(x)\cos\phi_1 + \\ + 2B_y q(x)\cos\phi_2 - Aq^2(x)\sin\phi_1 - Bq^2(x)\sin\phi_2]\, e^{iK\sigma(x)} .$$

The boundary value problem becomes:

$$L[u] = e^{iK\sigma}\Big[K^2\Big\{[n^2 - (\sigma_x)^2 - q^2](A\sin\phi_1 + B\sin\phi_2) + \\ + \frac{1}{\sin\phi_1}\frac{\partial}{\partial y}(A_y \sin^2\phi_1) + \frac{1}{\sin\phi_2}\frac{\partial}{\partial y}(B_y \sin^2\phi_2)\Big\} + \\ + iK[2\sigma_x(A\sin\phi_1 + B\sin\phi_2)_x + \sigma_{xx}(A\sin\phi_1 + B\sin\phi_2)] + \\ + (A\sin\phi_1 + B\sin\phi_2)_{xx}\Big] = 0 . \tag{2.10}$$

Substituting (2.8) into (2.7) we get

$$K^2[(A_y\sin\phi_1 + B_y\sin\phi_2) + q(A\cos\phi_1 + B\cos\phi_2) + \\ + iZ_jD_j(A\sin\phi_1 + B\sin\phi_2)] - iK\sigma_x\frac{dh_j}{dx}(A\sin\phi_1 + B\sin\phi_2) + \\ - \frac{dh_j}{dx}(A\sin\phi_1 + B\sin\phi_2)_x = 0 , \text{ on } y = h_j(x) \qquad j = 1, 2 . \tag{2.11}$$

We use (2.9) in (2.10) and (2.11), and equate separately to zero the coefficients of each power of K. Equating to zero the coefficient of K^2 in (2.10) yields

$$[n^2(x) - (\sigma_x)^2 - q^2(x)](A\sin\phi_1 + B\sin\phi_2) = 0 ,$$

since A_y and B_y are $O(K^{-1})$. Thus, since $(A\sin\phi_1 + B\sin\phi_2) \neq 0$,

$$(\sigma_x)^2 = n^2 - q^2 \equiv N^2 . \tag{2.12a}$$

From eq. (2.11) we get similarly

$$l_1 A_0 + f_1 B_0 = 0 \text{ for } y = h_1(x)$$

and

$$l_2 A_0 + f_2 B_0 = 0 \text{ for } y = h_2(x)$$

where

$$l_1 = q \cos\phi_1 + iZ_1 D_1 \sin\phi_1\,, \quad f_1 = q\cos\phi_2 + iZ_1 D_1 \sin\phi_2$$
$$l_2 = q \cos\phi_1 + iZ_2 D_2 \sin\phi_1\,, \quad f_2 = q\cos\phi_2 + iZ_2 D_2 \sin\phi_2\,.$$

Hence

$$\begin{pmatrix} l_1 & f_1 \\ l_2 & f_2 \end{pmatrix} \cdot \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = 0\,, \tag{2.13a}$$

The last relation implies that the determinant of the matrix is equal to zero:

$$M = \det \begin{pmatrix} l_1 & f_1 \\ l_2 & f_2 \end{pmatrix} = 0\,. \tag{2.13b}$$

The form of l_1, l_2, f_1, f_2 on the boundaries is seen to be:

on $y = h_1(x)$

$$l_1 = q(x)\,, \quad f_1(x) = q(x)\cos q(x)\gamma + iZ_1 D_1 \sin q(x)\gamma$$

on $y = h_2(x)$

$$l_2 = q(x)(h_2 - h_1) - iZ_2 D_2 \sin q(x)\gamma\,, \quad f_2 = q(x)\,,$$

where $\gamma(x) \equiv h_1(x) - h_2(x)$.

The determinant (2.13) may be solved, yielding

$$i\gamma(x)\,\frac{q^2 - Z_1 Z_2 D_1 D_2}{Z_1 D_1 - Z_2 D_2} = -[q(x)\gamma(x)]\cot[q(x)\gamma(x)]\,. \tag{2.13c}$$

Note that $D_j = 1 + O(K^{-2})$, compare with (2.3). Thus for purposes of calculating A_0, A_1, B_0, B_1 only, we may take in (2.13a) $D_j = 1$, $j = 1, 2$. Eq. (2.13c) is a transcendental equation for the determination of $q(x)$: D_j, Z_j, $h_j (j = 1, 2)$ are given functions of x. Solving (2.13c) numerically on a set of points $x = \{x_\nu\}$, we determine from it the given eigenvalues $q_i = q_i(x_\nu)$ $(i = 1, 2, 3 \ldots)$.

Once the eigenvalues q_i are found, they are substituted back into eq. (2.12), yielding a "modal eiconal equation" for each eigenvalue. The "equivalent refractive index" N is different for each mode. Obviously $N^2(x)$ is positive only for a finite set of eigenvalues q_i $(i = 1, 2, \ldots, m)$. These are the propagating modes. All other modes will yield an imaginary σ and correspond to evanescent modes.

The solution of (2.12) is standard. In two dimensions we get immediately:

$$\sigma_j(x) = \int_{x_0}^{x} [n^2(x) - q_j(x)]^{\frac{1}{2}} dx \qquad j = 1, 2, \ldots, m\,. \tag{2.12b}$$

In the special case when $h_1(x) = -h_2(x) > 0$, similar results were obtained [1]. However, (2.13c) is replaced by a simpler relation: $(\gamma h)\tan(\gamma h) = DZh$, where $h \equiv h_1$ and $\gamma(x) = \phi(x)/y$.

The equations for the determination of the coefficients A_m and B_m are obtained by equating to zero the coefficients of K^n $(n = 1, 0, -1, -2, \ldots)$. When we equate to zero the coefficient of K^1 in (2.10) we get:

$$\frac{1}{\sin\phi_1}\frac{\partial}{\partial y}(A_{1y}\sin^2\phi_1) = 2\sigma_x[A_0(x)\sin\phi_1]_x + \sigma_{xx}A_0(x)\sin\phi_1\,, \tag{2.14a}$$

$$\frac{1}{\sin\phi_2}\frac{\partial}{\partial y}(B_{1y}\sin^2\phi_2) = 2\sigma_x[B_0(x)\sin\phi_2]_x + \sigma_{xx}B_0(x)\sin\phi_2\,. \tag{2.14b}$$

The separation between A_m and B_m is possible since $L[u_1] = 0$ and $L[u_2] = 0$ separately (see eq. (2.8)).

Multiplying (2.14a) and (2.14b) by $\sin\phi_1$ and $\sin\phi_2$ respectively, and integrating (2.14a) from y to $h_1(x)$ and (2.14b) from $h_2(x)$ to y, we get:

$$A_{1y}(x, y) = \tfrac{1}{2}\sin^{-2}\phi_1 \left(\sigma_{xx}A_0 + 2\sigma_x A_{0x} + \sigma_x A_0 \frac{\partial}{\partial x}\right)\left[(h_1(x) - y) + \frac{\sin 2\phi_1}{2q(x)}\right] \tag{2.15a}$$

$$B_{1y}(x, y) = -\tfrac{1}{2}\sin^{-2}\phi_2 \left(\sigma_{xx}B_0 + 2\phi_x B_{0x} + \sigma_x B_0 \frac{\partial}{\partial x}\right)\left[(h_2(x) - y) + \frac{\sin 2\phi_2}{2q(x)}\right]. \tag{2.15b}$$

It can be easily seen (by L'Hospital's Rule) that A_{1y} is regular as $y \to h_1$, and B_{1y} is regular as $y \to h_2$. We integrate (2.15a, b) indefinitely in y, to obtain:

$$A_1(x, y) = F_0(x, y) + f_1(x) \tag{2.16a}$$

$$B_1(x, y) = G_0(x, y) + g_1(x) \tag{2.16b}$$

where $g_1(x)$ and $f_1(x)$ are still unknown functions in x. G_0 and F_0 are given by:

$$F_0(x, y) = \frac{1}{2q}[y - h_1(x)] \cdot \left[\sigma_{xx}A_0 + 2\sigma_x A_{0x} - \frac{q'}{q}\sigma_x A_0\right] \cot\phi_1 + + \sigma_x A_0 \left[\frac{q'}{2q}(y - h_1(x))^2 - h_{1x}(y - h_1(x))\right], \tag{2.17a}$$

$$G_0(x, y) = -\frac{1}{2q}[y - h_2(x)] \cdot \left[\sigma_{xx}B_0 + 2\sigma_x B_{0x} - \frac{q'}{q}\sigma_x B_0\right] \cot\phi_2 + + \sigma_x B_0 \left[\frac{q'}{2q}(y - h_2(x))^2 - h_{2x}(y - h_2(x))\right]. \tag{2.17b}$$

Hence A_1 and B_1 could be calculated if we know A_0, B_0, f_1, g_1 and $\sigma(x)$. Each eigenvalue q_j corresponds to a different σ_j and respectively to different A_{0j}, B_{0j}, A_{1j} and B_{1j}. The whole asymptotic solution up to some order in K will be obtained by summing all the propagating modes.

To obtain $A_0(x)$ and $B_0(x)$ we equate to zero the coefficient of K^1 in (2.11). We get:

$$B_{1y}(x, h_1) + B_1(x, h_1)[q\cot(q\gamma) + iZ_1D_1] + A_1(x, h_1)\frac{q}{\sin(q\gamma)} + \sigma_x h_{1x}B_0(x) = 0, \tag{2.18a}$$

$$A_{1y}(x, h_2) + A_1(x, h_2)[q\cot(-q\gamma) + iZ_2D_2] + B_1(x, h_2)\frac{q}{\sin(-q\gamma)} + \sigma_x h_{2x}A_0(x) = 0. \tag{2.18b}$$

Using Eq. (2.13a) we have an additional relation between A_0 and B_0:

$$\frac{A_0(x)}{B_0(x)} = -\cos(q\gamma) - \frac{iZ_1D_1}{q}\sin(q\gamma). \tag{2.19}$$

Now we substitute B_{1y}, B_1, A_{1y}, A_1, as given by (2.15) and (2.16), into (2.18a, b). To substitute A_{0x} and B_{0x} we take the x derivative in (2.19). Eq. (2.18), (2.19) and its derivative are in the form:

$$A_0(x) = \alpha(x)B_0(x) \quad \text{and} \quad A_0'(x) = \alpha'(x)B_0 + \alpha(x)B_0'(x)$$

$$\begin{aligned} a_1B_0 + b_1B_0' + c_1g_1 + d_1A_0 + e_1A_0' + r_1f_1 &= 0 \\ a_2B_0 + b_2B_0' + c_2g_1 + d_2A_0 + e_2A_0' + r_2f_1 &= 0 \end{aligned} \tag{2.20}$$

where a_j, b_j, c_j, d_j, e_j, r_j $(j=1, 2)$ are known functions of x [see details in Appendix A]. Using (2.13), Eqs. (2.20) are solvable and we get ordinary differential equation for $B_0(x)$:

$$-\frac{d}{dx}B_0(x) = B_0(x)D(x) \tag{2.21}$$

where $D(x)$ is given in Appendix A. Solving (2.21) we get from (2.19) $A_0(x)$. Thus $A_0(x)$ and

$B_0(x)$ are known. To find the further expansion coefficients A_m, B_m ($m=2, 2 \ldots$) we have to equate to zero the coefficients of K^{-m} ($m=0, 1, \ldots$) in (2.10), (2.11) and use A_{m-1}, A_{m-2} and B_{m-1}, B_{m-2}.

Once A_0 and B_0 are known we have B_1 and A_1 from eqs. (2.17a, b). However, the additive functions g_1 and f_1 are still unknown and have to found from the boundary conditions on A_2, B_2.

3. Dirichlet Boundary Condition

We define a similar problem to that of Section 2 but with the condition $u=0$, i.e. Dirichlet's conditions on the boundaries ∂R. The asymptotic expansion of an exact solution solved in Section 5 serves as a model for an Ansatz. We show that the choice of a suitable Ansatz is not obvious. We may be able to find the first term in the asymptotic series, but higher terms turn out not to be twice differentiable and therefore do not satisfy the P.D.E. itself, and constitute rather a weak solution. This may indicate a coupling between the modes, a well known phenomena in wave guides whenever the surfaces are not two parallel planes. For the sake of simplicity we take $H_2(x)\equiv 0$, and $n(x)=1$. We introduce the change of scale as in (2.5), and get

$$L[u] = \frac{\partial^2 u}{\partial x^2} + K^2 \left[\frac{\partial^2}{\partial y^2} + 1\right] u=0\,, \quad -\infty < x < \infty\,, \quad 0 \leqq y \leqq h(x)\,, \tag{3.1}$$

and

$$u(x, 0)=u(x, h(x))=0\,. \tag{3.2}$$

We assume first a solution of the form

$$u(x, y)=A(x, y; K) \sin\left(\frac{n\pi y}{h(x)}\right) e^{iK\sigma(x)} \qquad n=1, 2 \ldots \tag{3.3}$$

This form automatically satisfies the boundary conditions (3.2), thus we have only to take care of the P.D.E. (3.1).

We assume the same asymptotic expansion for A as in Section 2, namely:

$$A(x, y; K) \sim A_0(x) + \sum_{m=1}^{\infty} A_m(x, y)(iK)^{-m}\,, \tag{3.4}$$

with

$$A_j \in C^2\,, \qquad j=0, 1, 2, \ldots\,.$$

Using (3.3) and (3.4) in (3.1) we get a recursive set of differential equations:

$$\sigma_x^2(x)=1-\left(\frac{n\pi}{h(x)}\right)^2 \equiv N^2(x)\,, \qquad n=1, 2, \ldots p \tag{3.5}$$

$$\frac{1}{\sin\phi}\cdot\frac{\partial}{\partial y}(A_{1y}\sin^2\phi)=2\sigma_x(A_0\sin\phi)_x+\sigma_{xx}A_0\sin\phi \tag{3.6}$$

$$\frac{1}{\sin\phi}\cdot\frac{\partial}{\partial y}(A_{my}\sin^2\phi)=2\sigma_x(A_{m-1}\sin\phi)_x+\sigma_{xx}A_{m-1}\sin\phi+(A_{m-2}\sin\phi)_{xx} \tag{3.7}$$

where

$$\phi \equiv \frac{n\pi}{h(x)}y\,, \qquad m \geqq 2\,.$$

p is the number of propagating modes, i.e. for $1 \leqq n \leqq p$, $N(x)$ is real. Solution of equation (3.5) is standard (see eq. (2.12)). To solve (3.6) we note that (3.6) can be written as follows:

$$\frac{\partial}{\partial y}(A_{1y}\sin^2\phi) = \left(2\sigma_x A_{0x}+\sigma_{xx}A_0+\sigma_x A_0\frac{d}{dx}\right)\sin^2\phi\,. \tag{3.8}$$

This equation can be integrated with respect to y between $y=0$ and $y=h(x)$. Since $\sin\phi=0$

at both ends, we get:

$$2\sigma_x A_{0x} h + \sigma_{xx} A_0 h + \sigma_x A_0 h_x = 0 . \tag{3.9}$$

The solution of equation (3.9) is:

$$A_0(x) = A_0(x_0) \left[\frac{N(x_0) h(x_0)}{N(x) h(x)}\right]^{\frac{1}{2}}, \tag{3.10}$$

where $N(x)$ is defined by eq. (3.5).

The fact that our analysis fails near the cut off of a mode is seen by the fact that $N \to 0$, at such points. (In the problem of Section 2 this is not so obvious, and some calculations have to be done to see explicitly the manner in which A_0 tends to infinity in the vicinity of the cut off of a mode).

Having calculated A_0, we return to (3.6) whose right-hand side is now a known function. In order to calculate A_1 we rewrite (3.8) as follows

$$\frac{1}{\sin(ay)} \frac{\partial}{\partial y} (Z \sin^2(ay)) = F(x, y) . \tag{3.11}$$

The right-hand side of (3.11) is a known function, with $Z = A_{1y}$ and $a = n\pi/h(x)$.

Eq. (3.11) may be integrated directly;

$$Z = \frac{1}{\sin^2(ay)} \int_{y_0}^{y} F(y') \sin(ay') dy' . \tag{3.12}$$

From (3.6) we see that F is of the form $F = l \sin(ay) + ry \cos(ay)$ thus (3.12) can be calculated explicitly, yielding

$$Z = \frac{1}{\sin^2(ay)} \left\{ \frac{l}{2} \left[(y - y_0) - \frac{\sin(2ay)}{2a}\right] + \frac{r}{4a} \left[\frac{\sin(2ay)}{2a} - y \cos(2ay)\right]\right\}, \tag{3.13}$$

with l and r known functions of x.

For the m-th mode, the sine function becomes zero $(m+1)$ times in our domain. A_{1y} becomes infinitely large there. With the help of y_0, a constant of integration, we may exclude one "infinity point" from A_{1y}. All we can do here, is to build a piece-wise continuous A_{1y}, and A_1 will be piece wise differentiable. There are infinity many ways to do it. One way may be to choose the discontinuity points at the zeroes of $\cos(ay)$, that is halfways between the jth and the $(j+1)$th, $(j = 1, \ldots m+1)$ zero of $\sin \phi$.

$A_1(x, y)$ as obtained above violates the assumption that $A_j \in C^2 \, \forall j$. We may try, therefore, to modify our Ansatz (3.3) as follows:

$$u(x, y) = \quad A_0(x) \sin\left(\frac{n\pi y}{h(x)}\right) + A(x, y; K) \quad e^{iK\sigma(x)} \tag{3.14}$$

where $A_0(x)$ is given by (3.10) and A has the asymptotic expansion:

$$A(x, y; K) \sim \sum_{j=1}^{\infty} A_j(x, y)(iK)^{-j},$$

with

$$A_j \in C^2 \quad \text{and} \quad A_j(x, 0) = A_j(x, h(x)) = 0 . \tag{3.15}$$

The boundary conditions are satisfied automatically because of (3.15) and the recursive set of equations is:

$$\sigma_x^2 = 1 - \left(\frac{n\pi}{n(x)}\right)^2 = 1 - q^2(x), \tag{3.17a}$$

$$A_{1yy} + q^2 A_1 = 2\sigma_x (A_0 \sin \phi)_x + \sigma_{xx} A_0 \sin \phi , \tag{3.17b}$$

$$A_{2yy} + q^2 A_2 = -(2\sigma_x A_{1x} + \sigma_{xx} A_1) + (A_0 \sin \phi)_{xx} , \quad etc. \tag{3.17c}$$

Equations (3.17b) with boundary conditions (3.15) cannot be satisfied unless $A_1 \equiv 0$. This statement may be easily verified. The Ansatz (3.14) which is quite general is not suited well to solve the stated problem. The "weak" solution obtained from (3.13) indicates most probably that the modes (both propagating and evanescent) are coupled in such a way as to satisfy the specified boundary conditions. Since we expanded the solution in an asymptotic series for $K \to \infty$, the lowest order approximation is given by neglecting A_j for $j \geqq 1$. Up to this order of approximation the modes appear uncoupled.

4. Uniform Asymptotic Expansions in Thin Domains

The two solutions obtained in Sections 2 and 3 cease to exist in the vicinity of the cut off points of a mode. Therefore to obtain asymptotic solutions valid in the whole region under consideration, it is necessary to introduce uniform asymptotic expansions. (u. a. e.) [5, 6].

First we consider the Dirichlet problem (Section 3). Instead of (3.3), we assume a solution of the form:

$$u(x, y) = \sin\left(\frac{n\pi y}{h(x)}\right)\left\{B(x, y; K) A_i[-K^{\frac{2}{3}}\rho(x)] + K^{-\frac{1}{3}} C(x, y; K) A_i'[-K^{\frac{2}{3}}\rho(x)]\right\}, \quad (4.1)$$

where A_i is the Airy function which satisfies the differential equation

$$A_i''(x) - xA_i(x) = 0 . \quad (4.2)$$

A_i and A_i' are linearly independent.

We assume that B and C have the following asymptotic expansions:

$$B(x, y; K) \sim B_0(x) + \sum_{j=1}^{\infty} B_{2j}(x, y) K^{-2j} \quad (4.3a)$$

$$C(x, y; K) \sim \sum_{j=0}^{\infty} C_{2j+1}(x, y) K^{-(2j+1)} \quad (4.3b)$$

(We will discuss this choice of Ansatz subsequently).

We need the following derivatives: $(e' \equiv e_x,\ A_i' \equiv (\partial/\partial x) A_i)$

$$u_x = (B \sin\phi)_x A_i - K^{\frac{2}{3}} \rho_x B \sin\phi A_i' + K^{-\frac{1}{3}}(C \sin\phi)_x A_i' + K\rho\rho_x C \sin\phi A_i$$

$$\begin{aligned} u_{xx} = & -K^2\rho\rho_x^2 B \sin\phi A_i + 2K\rho\rho_x (C\sin\phi)_x A_i + K(\rho\rho_x)_x C \sin\phi A_i + (B\sin\phi)_{xx} A_i + \\ & - K^{\frac{5}{3}}\rho\rho_x^2 C \sin\phi A_i' - 2K^{\frac{2}{3}}\rho_x (B\sin\phi)_x A_i' - K^{\frac{2}{3}}\rho_{xx} B \sin\phi A_i' + \\ & + K^{-\frac{1}{3}}(C\sin\phi)_{xx} A_i' \end{aligned}$$

$$u_y = B_y \sin\phi A_i + \frac{n\pi}{h(x)} B \cos\phi A_i + K^{-\frac{1}{3}} C_y \sin\phi A_i' + K^{-\frac{1}{3}} \frac{n\pi}{h(x)} C \cos\phi A_i'$$

$$\begin{aligned} u_{yy} = & \left[B_{yy} \sin\phi + 2\frac{n\pi}{h(x)} B_y \cos\phi - \left(\frac{n\pi}{h}\right)^2 B \sin\phi\right] A_i + \\ & + K^{-\frac{1}{3}}\left[C_{yy} \sin\phi + 2\frac{n\pi}{h(x)} C_y \cos\phi - \left(\frac{n\pi}{h}\right)^2 C \sin\phi\right] A_i' \end{aligned}$$

where

$$\phi \equiv \frac{n\pi}{h(x)} y .$$

Hence the operator takes the form:

$$L[u] = A_i \left[K^2 \left\{ \left[1 - \left(\frac{n\pi}{h}\right)^2 - \rho\rho_x^2 \right] \cdot B \sin\phi + \frac{1}{\sin\phi} \frac{\partial}{\partial y} (B_y \sin^2\phi) \right\} \right.$$
$$\left. + K[2\rho\rho_x (C \sin\phi)_x + (\rho\rho_x)_x C \sin\phi] + (B \sin\phi)_{xx} \right] +$$
$$+ K^{-\frac{1}{3}} A_i' \left[K^2 \left\{ \left[1 - \left(\frac{n\pi}{h}\right)^2 - \rho\rho_x^2 \right] C \sin\phi + \frac{1}{\sin\phi} \frac{\partial}{\partial y} (C_y \sin^2\phi) \right\} \right.$$
$$\left. - K[2\rho_x (B \sin\phi)_x + \rho_{xx} B \sin\phi] + (C \sin\phi)_{xx} \right] = 0\,. \tag{4.4}$$

Using (4.3a, b) in (4.4) and equating to zero coefficients of powers of K, we get a recursive set of equations:

$$\rho\rho_x^2 = 1 - \left(\frac{n\pi}{h(x)}\right)^2 \equiv N_n^2(x)\,, \tag{4.5}$$

$$\frac{1}{\sin\phi} \frac{\partial}{\partial y} (C_{1y} \sin^2\phi) = 2\rho_x (B_0 \sin\phi)_x + \rho_{xx} B_0 \sin\phi\,, \tag{4.6}$$

$$\vdots$$

etc.

The solution of (4.5) is:

$$\rho_n(x) = \left[\frac{3}{2} \int_{x_0}^{x} \left\{ 1 - \left[\frac{n\pi}{h(x')} \right]^2 \right\}^{\frac{1}{2}} dx' \right]^{\frac{2}{3}}, \qquad n = 1, \dots p\,. \tag{4.7}$$

Again, p is the number of propagating modes, i.e. for $1 \leqq n \leqq p$ $\rho_n(x)$ is real. Although $\rho_n(x)$ becomes zero at cut-off point of the n-th mode (x_n^c), $\rho_{nx}(x_n^c) \neq 0$. Indeed we have at $x = x_n^c$

$$\frac{d\rho_n}{dx} = \left(\frac{dN^2}{dx}\right)^{\frac{1}{3}},$$
$$\frac{d^2\rho_n}{dx^2} = \frac{3}{5} \frac{d}{dx} \left(\frac{dN^2}{dx}\right)^{\frac{1}{3}}. \tag{4.8}$$

x_n^c may be obtained by solving the equation

$$N_n(x) = 0 \;\text{ or }\; h(x) = n\pi\,. \tag{4.9}$$

The relations (4.8) follow immediately by expanding $\rho(x)$ in equation (4.5) in a power series about the cut-off point and using $\rho_n(x_n^c) = 0$. We see that although equation (4.5) is analogous to the eiconal equation (3.5), there is an important difference: unlike $\sigma_x(x)$, ρ_x does not vanish at cut-off points, indicating a regular behaviour of $B_0(x)$ there. Indeed, to get $B_0(x)$ we multiply (4.6) by $\sin\phi$ and integrate it from $y=0$ to $y=h(x)$. Since $\sin\phi = 0$ at $y=h$ we get

$$0 = 2\rho_x B_{0x} h + \rho_{xx} B_0 h + \rho_x B_0 h_x\,, \tag{4.10}$$

therefore

$$B_0(x) = B_0(x_0) \left[\frac{h(x_0)\rho_x(x_0)}{h(x)\rho_x(x)} \right]^{\frac{1}{2}}. \tag{4.11}$$

Thus, by (4.8), $B_0(x)$ is regular at the cut-off point. Far away from such points the first term of the uniform asymptotic solution behaves like the first term of the ordinary asymptotic solution. One may see this by taking the first term in the asymptotic expansion of the Airy function far from the cut-off point. This will be shown in Appendix B and explains why $C_0(x)$ has to be taken identically equal to zero. As a consequence the specific form (4, 3a, b) of the asymptotic expansion is assumed.

The next term in the expansion is $C_1(x, y)$. We see that eq. (4.6) is exactly the same as eq. (3.6) and we shall obtain the same kind of "weak" solution. We may try to change the Ansatz (4.1) to be:

$$u = B(x, y, K) A_i(-K^{\frac{2}{3}}\rho) + K^{-\frac{1}{3}} \cdot C(x, y, K) A_i'(-K^{\frac{2}{3}}\rho) . \tag{4.12}$$

We assume again that B and C have the same asymptotic expansions (4.3a) and (4.3b) with

$$C_{2j+1}(0, x) = C_{2j+1}(h, x) = B_{2j}(0, x) = B_{2j}(h, x) = 0 . \tag{4.13}$$

$B_0(x)$ is assumed to be given by (4.11).

Thus (4.4) takes now the form:

$$\begin{aligned} L[u] = A_i \Big[K^2 \Big\{ \Big[1 - \left(\frac{n\pi}{h}\right)^2 - \rho\rho_x^2 \Big] B_0 \sin\phi + (1-\rho\rho_x^2) B + B_{yy} \Big\} \\ + K\{2\rho\rho_x C_x + (\rho\rho_x)_x C\} + (B_0 \sin\phi + B)_{xx} \Big] + \\ + K^{-\frac{1}{3}} A_i' [K^2 \{C_{yy} + (1-\rho\rho_x^2) C\} - K\{2\rho_x (B_0 \sin\phi + B)_x \\ + \rho_{xx}(B_0 \sin\phi + B)\} + C_{xx}] = 0 . \end{aligned} \tag{4.14}$$

The recursive set of equations is

$$\begin{aligned} \rho\rho_x^2 = 1 - \left(\frac{n\pi}{n(x)}\right)^2 = 1 - q^2(x) , \\ C_{1yy} + q^2(x) C_1 = 2\rho_x (B_0 \sin\phi)_x + \rho_{xx}(B_0 \sin\phi) , \\ B_{2yy} + q^2(x) B_2 = -2\rho\rho_x C_{1x} - (\rho\rho_x)_x C_1 - (B_0 \sin\phi)_{xx} , \\ \vdots \\ etc. \end{aligned} \tag{4.15}$$

The equation for C_1 has the same form as that of (3.17b) therefore we encounter here the same problem as in Section 3: $C_1(x, y) \equiv 0$, and from (4.6) we may obtain a "weak" uniform solution. The problem indicates once again coupling between modes in order to satisfy the specified boundary conditions.

Taking $\cos\phi$ instead of $\sin\phi$ we may deal in exactly the same way with Neumann boundary conditions.

The "impedance" boundary conditions problem (eqs. (2.1) (2.2)) can also be solved by a uniform asymptotic expansion.

The proposed form of the solution is:

$$\begin{aligned} u = \sin\phi_1 [B A_i(-K^{\frac{2}{3}}\rho) + K^{-\frac{1}{3}} \cdot C A_i'(-K^{\frac{2}{3}}\rho)] \\ + \sin\phi_2 [D A_i(-K^{\frac{2}{3}}\rho) + K^{-\frac{1}{3}} \cdot E A_i'(-K^{\frac{2}{3}}\rho)] , \end{aligned} \tag{4.16}$$

hence

$$u = [B \sin\phi_1 + D \sin\phi_2] A_i(-K^{\frac{2}{3}}\rho) + K^{-\frac{1}{3}} [C \sin\phi_1 + E \sin\phi_2] A_i'(-K^{\frac{2}{3}}\rho) ,$$

where

$$\phi_1(x, y) \equiv q(x)(y - h_1(x)) , \quad \phi_2(x, y) \equiv q(x)(y - h_2(x)) .$$

We assume that B, C, D, E have the following asymptotic expansions:

$$\begin{aligned} B(x, y; K) \sim B_0(x) + \sum_{j=1}^{\infty} B_{2j}(x, y) K^{-2j} ; \\ D(x, y; K) \sim D_0(x) + \sum_{j=0}^{\infty} D_{2j}(x, y) K^{-2j} \\ C(x, y; K) \sim \sum_{j=0}^{\infty} C_{2j+1}(x, y) K^{-(2j+1)} \\ E(x, y; K) \sim \sum_{j=0}^{\infty} E_{2j+1}(x, y) K^{-(2j+1)} . \end{aligned} \tag{4.17}$$

Substituting (4.16) into the operator

$$\left[\frac{\partial^2}{\partial x^2} + K^2 \frac{\partial^2}{\partial y^2} + K^2\right] u = 0,$$

we have:

$$L[u] = A_i \Big\{K^2 \Big[(1-q^2-\rho\rho_x^2)(B \sin\phi_2 + D \sin\phi_2) + \frac{1}{\sin\phi_1}\frac{\partial}{\partial y} + (C_y \sin\phi_1)$$
$$+ \frac{1}{\sin\phi_2}\frac{\partial}{\partial y}(D_y \sin\phi_2)\Big] + K[2\rho\rho_x(C\sin\phi_1 + E\sin\phi_2)_x +$$
$$+ (\rho\rho_x)_x(C\sin\phi_1 + E\sin\phi_2)] + (B\sin\phi_1 + D\sin\phi_2)_{xx}\Big\} +$$
$$+ K^{-\frac{1}{3}}A_i' \Big\{K^2 \Big[(1-q^2-\rho\rho_x^2)(C\sin\phi_1 + E\sin\phi_2) + \frac{1}{\sin\phi_1}\frac{\partial}{\partial y}(C_y \sin^2\phi_1)$$
$$+ \frac{1}{\sin\phi_2}\frac{\partial}{\partial y}(E_y \sin^2\phi_2)\Big] - K[2\rho_x(B\sin\phi_2 + D\sin\phi_2)_x$$
$$+ \rho_{xx}(B\sin\phi_1 + D\sin\phi_2)] + (C\sin\phi_1 + E\sin\phi_2)_{xx}\Big\} = 0\,.$$

Using (4.17) we get a recursive set of equations:

$$\begin{aligned}
&\rho\rho_x^2 = 1 - q^2(x)\,,\\
&\frac{1}{\sin\phi_1}\frac{\partial}{\partial y}(C_{1y}\sin^2\phi_1) = 2\rho_x(B_0\sin\phi_1)_x + \rho_{xx}B_0\sin\phi_1\,,\\
&\frac{1}{\sin\phi_2}\frac{\partial}{\partial y}(E_{1y}\sin^2\phi_2) = 2\rho_x(D_0\sin\phi_2)_x + \rho_{xx}D_0\sin\phi_2\,,\\
&\frac{1}{\sin\phi_1}\frac{\partial}{\partial y}(B_{2y}\sin^2\phi_1) = -2\rho\rho_x(C_1\sin\phi_1)_x - (\rho\rho_x)_x C_1\sin\phi_1 - B_0\sin\phi_1\,,\\
&\frac{1}{\sin\phi_2}\frac{\partial}{\partial y}(D_{2y}\sin^2\phi_2) = -2\rho\rho_x(E_1\sin\phi_2)_x - (\rho\rho_x)_x E_1\sin\phi_2 - D_0\sin\phi_2\,,\\
&\vdots\\
&etc.
\end{aligned} \tag{4.18}$$

Substituting (4.16) in the boundary conditions (2.7) we get:

$$\begin{aligned}
&A_i(-K^{\frac{2}{3}}\rho)\Big\{K^2[(B_y\sin\phi_1 + D_y\sin\phi_2 + Bq\cos\phi_1 + Dq\cos\phi_2) +\\
&\quad + iZ_j\Phi_j(B\sin\phi_1 + D\sin\phi_2)] - K\rho\rho_x\frac{dh_j}{dx}(C\sin\phi_1 + E\sin\phi_2) +\\
&\quad - \frac{dh_j}{dx}(B\sin\phi_1 + D\sin\phi_2)_x\Big\} +\\
&+ A_i'(-K^{\frac{2}{3}}\rho)K^{-\frac{1}{3}}\Big\{K^2[(C_y\sin\phi_1 + E_y\sin\phi_2 + Cq\cos\phi_1 + Eq\cos\phi_2) +\\
&\quad + iZ_j\Phi_j(C\sin\phi_1 + E\sin\phi_2)] + K\rho_x\frac{dh_j}{dx}(B\sin\phi_1 + D\sin\phi_2) +\\
&\quad - \frac{dh_j}{dx}(C\sin\phi_1 + E\sin\phi_2)_x\Big\} = 0 \qquad j = 1, 2
\end{aligned} \tag{4.19}$$

where

$$\Phi_j(x) \equiv \left[1 + \left(\frac{\partial H_j}{\partial x}\right)^2\right]^{\frac{1}{2}}$$

has been used instead of D_j in (2.2) ($j = 1, 2$) to prevent confusion with $D(x, y; K)$ used here.

Using (4.17) we get a recursive set of equations. Equating to zero the coefficient of K^2 we get:

$$\begin{pmatrix} M \end{pmatrix} \cdot \begin{pmatrix} B_0 \\ D_0 \end{pmatrix} = 0 \text{ where } M = \begin{pmatrix} 1, & \cos q\gamma + i\phi_1 Z_1 \dfrac{\sin q\gamma}{q} \\ \cos q\gamma - i\phi_2 Z_2 \dfrac{\sin q\gamma}{q}, & 1 \end{pmatrix} \tag{4.20}$$

and $\gamma \equiv h_1 - h_2$.

Thus $\det(M)=0$, which leads to

$$-[\gamma(x)q(x)]\cdot[\cot(q\gamma)] = i\gamma \frac{q^2 - \Phi_1 \Phi_2 Z_1 Z_2}{\Phi_1 Z_1 - \Phi_2 Z_2} \tag{4.21}$$

exactly as in (2.14).

For lower powers of K we have

$$E_{1y} \sin q\gamma + C_1 q + E_1 q \cos q\gamma + i(Z_1 \Phi_1 E_1 \sin q\gamma) + \rho_x \frac{dh_1}{dx} D_0 \sin q\gamma = 0 \quad \text{on } y = h_1(x)$$

$$-C_{1y} \sin q\gamma + C_1 q \cos q\gamma + E_1 q - i(Z_2 \Phi_2 C_1 \sin q\gamma) - \rho_x \frac{dh_2}{dx} B_0 \sin q\gamma = 0 \quad \text{on } y = h_2(x). \tag{4.22}$$

A comparison of the first two equations in (4.18) with (2.14a, b) and equations (4.22) with (2.18a, b) show that these equations have exactly the same form, and the solution obtained there may be immediately used here. The difference is that in the uniform expansion formulas ρ_x appears, which is finite at cut-off points, while in the ordinary asymptotic expansion σ_x appears, which become zero at these points. Thus, our expansion is indeed uniform in the sense that it is regular across the turning point (or cut off point).

It may be interesting to make some remarks about the structure of our Ansatz (4.16) and (4.12). In general both have the same structure; the Airy function is multiplied by a series of even powers of K plus its derivative multiplied by a series of odd powers of K. For problems in thin domains, this structure seems to be suited well to remove the singularities of the ordinary asymptotic expansion. This approach could in all probability be used for vector Helmholtz equation as well. Such equations appear in problems of electromagnetic and elastic wave propagation.

Comparing Ansatz (2.8) with (4.16) we see at once that $A_0 e^{iK\sigma}$ may be compared with $B_0 A_i$-$(-K^{\frac{2}{3}}\rho)$, while $A_1 e^{iK\sigma}$ has to be compared with $C_1 \cdot A_i'(-K^{\frac{2}{3}}\rho)$ and so on. Eq. (4.18) demonstrates the coupling that exists between the coefficients B_m and C_m (and also D_m and E_m).

5. The Canonical Problem

By a "canonical problem" we understand a simple problem belonging to the class of problems we are dealing with, to which an exact solution can be found. We shall compare the asymptotic expansion of the exact solution with the expansions we obtained in Sections 3, 4. This comparison will serve as a check for the Ansatz we use for all the problems belonging to the same class and to which we do not have exact solutions. As a matter of fact, the asymptotic expansion of exact solutions often serve as a motivation for the Ansatz used for wider classes of problems.

We choose the following canonical problem:

$$\begin{aligned} &(\nabla^2 + K^2)u = 0 \text{ in } D \\ &u = 0 \text{ on } \partial D \end{aligned} \tag{5.1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial Y^2$, and the region D is defined by $0 < x < \infty$, $0 \leqq Y \leqq H(x)$, with $H(x) = \alpha x$, $\alpha > 0$ (see Fig. 2 on page 104). According to our assumptions of "thin domain" $\alpha = \alpha'/K$, $\alpha' = \text{const.}$

The exact solution is

$$u = AH_\nu^{(1)}(Kr)\sin(\nu\theta) \tag{5.2}$$

where θ_0 is measured in radians, $\tan\theta_0 = \alpha$, and $\nu = n\pi/\theta_0$.

The asymptotic expansion we are looking for is to be valid far from the origin. Since we are working in thin domains the angle of inclination has to be small. Therefore we may assume:

$$\theta_0 \cong \alpha \text{ and } x \gg 1 \text{ hence } r \cong x\,. \tag{5.4}$$

The first term in the asymptotic expansion of $H_\nu^{(1)}(x)$ for $(x-\nu)^3 \gg x$ is [8]

$$H_\nu^{(1)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left(1-\frac{\nu}{x^2}\right)^{-\frac{1}{4}}\cdot \exp i\left\{\left[\left(1-\frac{\nu^2}{x^2}\right)^{\frac{1}{2}}-\frac{\nu}{x}\cdot\cos^{-1}\frac{\nu}{x}\right]-\frac{\pi}{4}\right\}+O\left(\frac{1}{x}\right). \tag{5.5}$$

For $(x-\nu)^3 = O(x)$ a uniform asymptotic expansion has to be used. θ is real and positive hence: $\cos^{-1} y = -i\cosh y^{-1} = -i\ln\left[y+(y^2-1)^{\frac{1}{2}}\right]$. Thus

$$H_\nu(Kx) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\exp\left(-i\frac{\pi}{4}\right)(Kx)^{-\frac{1}{2}}\left[1-\frac{\nu^2}{(Kx)^2}\right]^{-\frac{1}{4}}$$

$$\times\left[\frac{\nu}{Kx}+\left(\frac{\nu^2}{(Kx)^2}-1\right)^{\frac{1}{2}}\right]^{-\nu}\cdot\exp\left\{iKx\left[1-\left(\frac{\nu}{Kx}\right)^2\right]^{\frac{1}{2}}\right\}+O\left(\frac{1}{Kx}\right). \tag{5.6}$$

Using the results of Section 3 we compute the first term in the expansion for the problem defined by (5.1). On the boundary $Y = H(x) = \alpha x$ becomes $y = h(x) = K\cdot\alpha x$. Hence by (3.5) we have

$$\sigma(x) = \int^x N(x')dx' = \frac{1}{K}\left\{\left[(Kx)^2-\nu^2\right]^{\frac{1}{2}}-\nu^2\tan^{-1}\left[\left(\frac{Kx}{\nu}\right)^2-1\right]^{\frac{1}{2}}\right\} \tag{5.7}$$

where

$$N = \left[1-\left(\frac{n\pi}{K\alpha x}\right)^2\right]^{\frac{1}{2}} = \left[1-\left(\frac{\nu}{Kx}\right)^2\right]^{\frac{1}{2}}.$$

To get $A_0(x)$, we use (3.10) and obtain:

$$A_0(x) = \text{const.}\,(Kx)^{-\frac{1}{2}}\left[1-\left(\frac{\nu}{Kx}\right)^2\right]^{-\frac{1}{4}}. \tag{5.8}$$

Since, by (5.4)

$$\theta \cong \frac{y}{Kx} \tag{5.9}$$

we get

$$\sin\frac{n\pi y}{h(x)} \cong \sin\frac{n\pi\theta}{\theta_0}. \tag{5.10}$$

For a more compact expression of $e^{iK\sigma(x)}$ we use the identity

$$e^{-i\nu\tan^{-1}\left[\left(\frac{(Kx)^2}{\nu}-1\right)^{\frac{1}{2}}\right]} = \left[\frac{\nu}{Kx}+\left(\left(\frac{\nu}{Kx}\right)^2-1\right)^{\frac{1}{2}}\right]^{-\nu}$$

thus

$$u \sim \left[A_0(x)+O\left(\frac{1}{K}\right)\right]\cdot\sin\left(\frac{n\pi\theta}{\theta_0}\right)$$

$$\times\exp\left[i\,Kx\left(1-\frac{\nu^2}{(Kx)^2}\right)^{\frac{1}{2}}\right]\cdot\left\{\frac{\nu}{Kx}+\left[\left(\frac{\nu}{Kx}\right)^2-1\right]^{-\frac{1}{2}}\right\}^{-\nu}. \tag{5.11}$$

Choosing the constant in

$A_0(x)$ as $\frac{2}{\pi}\exp\left(-i\frac{\pi}{4}\right)$

we see immediately the identity of (5.11) with (5.6) multiplied by $\sin(n\pi\theta/\theta_0)$. Hence we established the asymptotic correspondence of the first term in the exact solution with the first term in our Ansatz.

The same correspondence can be found between the first term of the uniform asymptotic expansion of the exact solution (5.2) and the first term calculated by the method of Section 4. We shall give the details in Appendix B.

6. Numerical Results

We demonstrate the method presented in this work, with a numerical example. For the sake of simplicity we solve a problem with Dirichlet's boundary conditions. The domain with $y=0$ and $y=h(x)$, where

$$h(x) = \begin{cases} 10 & \text{for } x \leqq 0 \\ 10\, e^{-x^2} & \text{for } 0 \leqq x \leqq 1 \\ 10/e & \text{for } x \geqq 1 \end{cases} \tag{6.1}$$

Using the formulas obtained in Sections 3 and 4, we compute the first term in the ordinary asymptotic expansion and in the uniform asymptotic expansion in the interval (0.1) and compare their behaviour. From (3.5) it is easily seen that there are at most three propagating modes. The second and the third mode have their turning points in the discussed interval. The turning point of each mode is computed by equating to zero the right hand side of (3.5). $A_0(x)$ as given

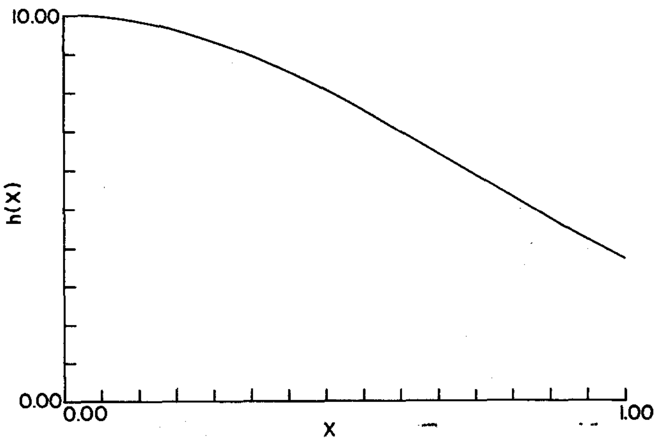

Figure A. The surface $y=h(x)$ (Equation (6.1)).

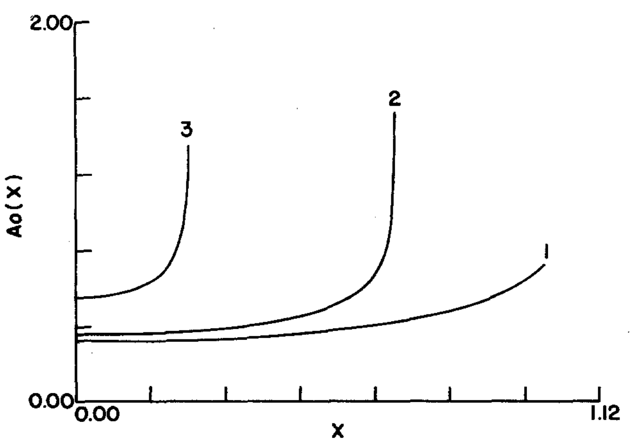

Figure B. The zero order coefficient $A_0(x)$ of the propagating modes (ordinary expansion). The dependence on $\sin\phi$ is not shown in the following figures.

by (3.10) breaks down at the turning points while the first term of the uniform expansion behaves smoothly.

Since the terms of the asymptotic expansion are defined up to a multiplicative constant, the figures presented have different y scales.

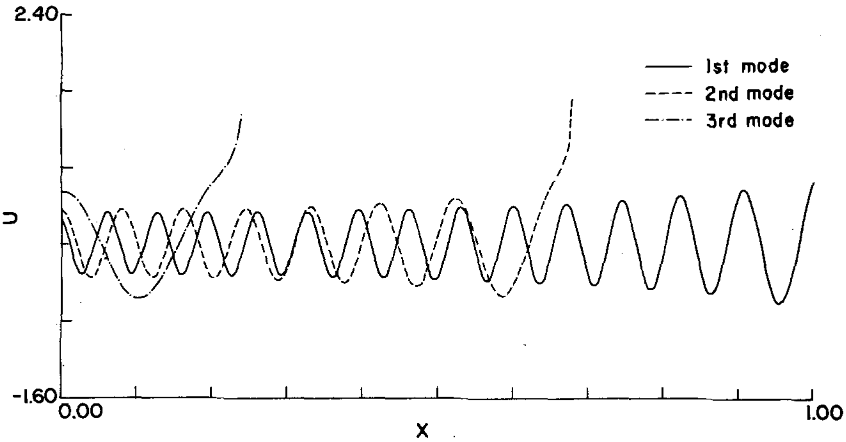

Figure C. The zero order approximation of the three propagating modes (ordinary expansion).

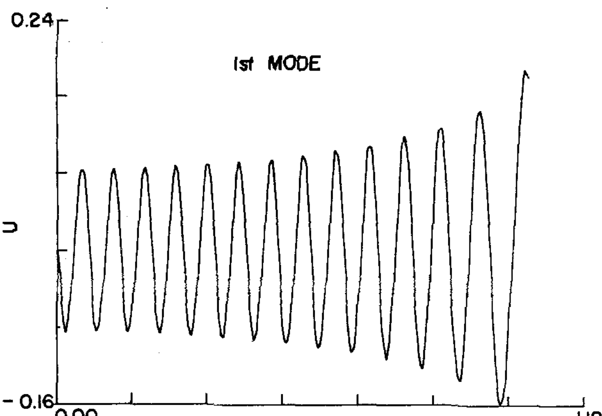

Figure D. The zero order approximation of the 1st propagating mode (uniform expansion).

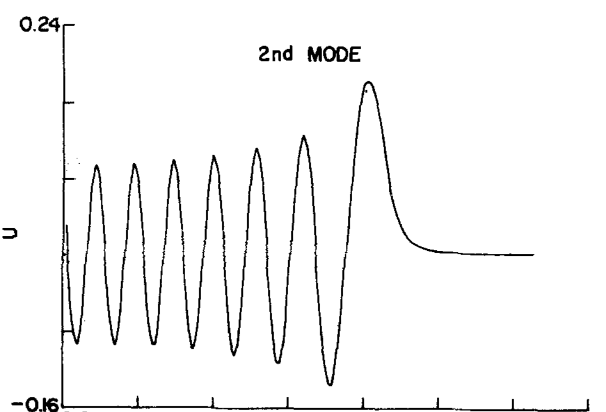

Figure E. The zero order approximation of the 2nd propagating mode (uniform expansion).

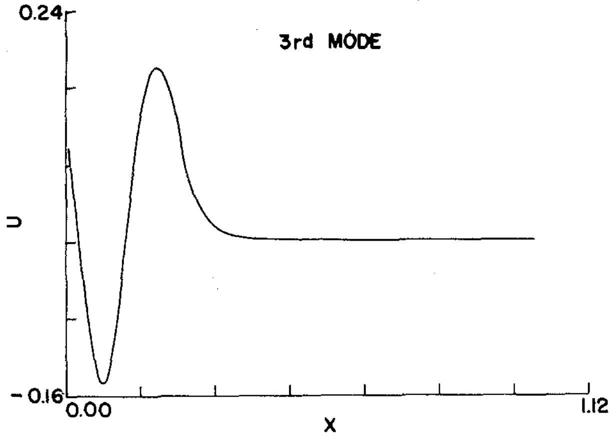

Figure F. The zero order approximation of the 3rd propagating mode (uniform expansion).

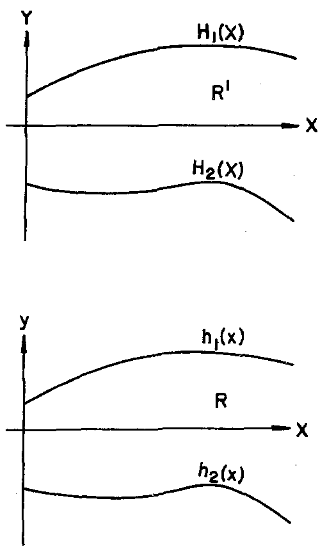

Figure 1.

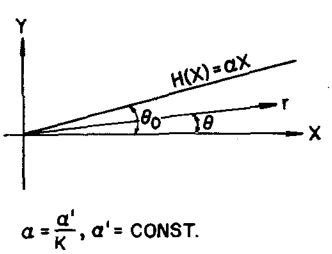

Figure 2.

Appendix A

We calculate now explicitly $A_0(x)$ and $B_0(x)$ of Section 2. From eqs. (2.19a, b) we have

$$\frac{A_0(x)}{B_0(x)} = \alpha(x) = -\cos(q\gamma) - \frac{iZ_1 D_1}{q}\sin(q\gamma), \tag{A1}$$

We can solve the set of equations (2.20) observing that

$$\det\begin{pmatrix} r_1, c_1 \\ r_2, c_2 \end{pmatrix} = 0. \tag{A2}$$

One may see this by comparing it with the matrix M in (2.13). We obtain:

$$D(x)B_0(x) + B_0'(x) = 0 \tag{A3}$$

where

$$D(x) = \frac{(a_1 + d_1\alpha + \alpha' e_1)C_2 - C_1(a_2 + d_2\alpha + \alpha' e_2)}{(b_1 + \alpha e_1)C_2 - C_1(b_2 + \alpha e_2)}, \tag{A4}$$

and

$\alpha' \equiv \frac{d}{dx}\alpha(x)$. Hence

$$B_0(x) = \exp\left[-\int_{x_0}^{x} D(x')dx'\right]. \tag{A5}$$

The coefficients a_i, b_i etc. which appear in eq. (2.20) are

$$a_1(x) = \sigma' h_1' - \frac{1}{2\sin^2(q\gamma)}\left[\sigma'' + \sigma' h_2' + \frac{\sigma'}{q}(q' h_1 - (h_2 q)')\cos(2q\gamma) + \right.$$
$$\left. - \frac{\sigma' q'}{2q^2}\sin(2q\gamma)\right] - \gamma(q\cot(q\gamma) - iZ_1 D_1)\left[\frac{(\sigma'' q - q'\sigma')}{2q^2}\cot(q\gamma) + \right.$$
$$\left. + \sigma'\left(\frac{q'}{2q}\gamma - h_2'\right)\right] \tag{A6}$$

$$b_1(x) = \frac{-\sigma'}{\sin^2(q\gamma)} - \frac{\sigma'\gamma}{q}[q\cot(q\gamma) + iZ_1 D_1]\cot(q\gamma) \tag{A7}$$

$$c_1(x) = q\cot(q\gamma) + iZ_1 D_1 \tag{A8}$$

$$d_1(x) = \frac{1}{2q}\left(\sigma'' - \frac{q'}{q}\sigma'\right)\frac{1}{\sin(q\gamma)} \tag{A9}$$

$$e_1(x) = \frac{\sigma'}{q\sin(q\gamma)} \tag{A10}$$

$$a_2(x) = -d_1(x) \tag{A11}$$

$$b_2(x) = -e_1(x) \tag{A12}$$

$$c_2(x) = -\frac{q}{\sin(q\gamma)} \tag{A13}$$

$$d_2(x) = \sigma' h_2' - \frac{1}{2\sin^2(q\gamma)}\left[\sigma'' + \sigma' h_1' + \frac{\sigma'}{q}(h_2 - (h_1 q)')\cos(2q\gamma) + \frac{\sigma' q'}{2q^2}\sin(2q\gamma)\right]$$
$$+ \gamma(-q\cot(q\gamma) + iZ_2 D_2)\left[\frac{-\sigma'' q + q'\sigma'}{2q^2}\cot(q\gamma) - \sigma'\left(\frac{q'}{2q}\gamma + h_1'\right)\right] \tag{A14}$$

$$e_2(x) = \frac{-\sigma'}{\sin^2(q\gamma)} - \frac{\sigma' y}{q}\left[-q\cot(q\gamma) + iZ_2 D_2\right]\cot(q\gamma) \tag{A15}$$

$$r_1(x) = \frac{q}{\sin(q\gamma)} \tag{A16}$$

$$r_2(x) = -q\cot(q\gamma) + iZ_2 D_2 . \tag{A17}$$

Appendix B

We showed in Section 5 that the asymptotic expansion of the exact solution of our canonical problem agrees with the asymptotic solution of that problem. In this appendix we shall show that the uniform asymptotic expansion of the exact solution agrees with the uniform asymptotic procedure of Section 5. The exact solution exists in the whole domain but, as already stated, whenever we are in the neighborhood of the turning point, our ordinary expansion breaks down and we have to use a uniform asymptotic expansion. The (u.a.e.) of $H_\nu^{(1)}$ is [8]:

$$H_\nu^{(1)}(\nu z) \sim 2e^{-\pi i/3}\left(\frac{4\xi}{1-z^2}\right)^{\frac{1}{4}}\left\{\frac{A_i(e^{2\pi i/3}\nu^{\frac{2}{3}}\xi)}{\nu^{\frac{1}{3}}} + \dots\right\}, \tag{B.1}$$

where

$$\tfrac{2}{3}\xi^{\frac{3}{2}} = \ln\frac{1+(1-z^2)^{\frac{1}{2}}}{z} - (1-z^2)^{\frac{1}{2}}, \tag{B.2a}$$

$$\tfrac{2}{3}(-\xi)^{\frac{3}{2}} = (z^2-1)^{\frac{1}{2}} - \cos^{-1}\left(\frac{1}{z}\right), \tag{B.2b}$$

and the branches being chosen so that ξ is real when $z>0$.

We take $z=Kx/\nu$ and it is easily seen that for (B.2a) we get

$$\xi = -\left\{\frac{3}{2}\left[\left(\left(\frac{Kx}{\nu}\right)^2 - 1\right)^{\frac{1}{2}} - \tan^{-1}\left(\left(\frac{Kx}{\nu}\right)^2 - 1\right)^{\frac{1}{2}}\right]\right\}^{\frac{2}{3}}. \tag{B.3}$$

For (B.1) we have

$$H_\nu^{(1)}(Kx) \sim 2e^{-\pi i/3}\left(\frac{4\xi\nu^2}{\nu^2-(Kx)^2}\right)^{\frac{1}{4}}\left\{\frac{A_i(e^{i2\pi/3}\nu^{\frac{2}{3}}\xi)}{\nu^{\frac{1}{3}}} + \dots\right\}. \tag{B.4}$$

Applying now the results of Section 4 we calculate explicitly the first term in the uniform expansion. From (4.8) we get:

$$\begin{aligned}\rho(x) &= \left\{\frac{3}{2}\int^x\left[1 - \left(\frac{n\pi}{h(x')}\right)^2\right]^{\frac{1}{2}} dx'\right\}^{\frac{2}{3}} \\ &= \left\{\frac{3}{2K}\left[((Kx)^2-\nu^2)^{\frac{1}{2}} - \nu\tan^{-1}\left(\left(\frac{Kx}{\nu}\right)^2 - 1\right)^{\frac{1}{2}}\right]\right\}^{\frac{2}{3}}.\end{aligned} \tag{B.5}$$

From (4.11) we get:

$$B_0(x) = [h(x)\rho_x(x)]^{-\frac{1}{2}} = \left[\frac{\rho(x)}{h^2(x)-(n\pi)^2}\right]^{\frac{1}{4}} = \frac{1}{\alpha^{\frac{1}{2}}}\left[\frac{\rho(x)}{(Kx)^2-\nu^2}\right]^{\frac{1}{4}}. \tag{B.6}$$

Therefore, the first order approximation is

$$u(x, y; h) \sim \sin\left(\frac{n\pi}{\alpha}\theta\right) B_0(x) A_i(-K^{\frac{2}{3}}\rho) \tag{B.7}$$

where $\rho(x)$ and $B_0(x)$ are given by (B.5) and (B.6).

Comparing (B.3) and (B.5) we have:

$$\xi = -\left(\frac{K}{\nu}\right)^{\frac{2}{3}}\rho(x). \tag{B.8}$$

Using this in (B.4) we get

$$H_\nu^{(1)}(Kx) \sim 2 \cdot 2^{\frac{1}{2}} \cdot e^{-\pi i/3} \left[\frac{\rho(x)}{(Kx)^2 - \nu^2} \right]^{\frac{1}{4}} \left\{ K^{\frac{1}{6}} A_i(-e^{2\pi i/3} K^{\frac{2}{3}} \rho) + \dots \right\}. \tag{B.9}$$

Apart from a constant we are now able to identify (B.7) with (B.9), multiplied by sin $(\theta n\pi/\alpha)$. Hence we established the equivalence of the first term in the asymptotic expansion of the exact solution with the first term of the uniform Ansatz.

The term $K^{\frac{1}{6}}$ appearing in (B.9) requires explanation: It arises from the fact that the uniform expansion of the Hankel function (B.1) has $A_i(-e^{2\pi i/3} K^{\frac{2}{3}} \rho)$, while we used in the Ansatz (4.1) or (4.12) $A_i(-K^{\frac{2}{3}} \rho)$.

Far from the turning point, the first term of our uniform-Ansatz (4.3) reduces to the first term of the ordinary-Ansatz. Recalling that

$$A_i(-\eta) \sim \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{4}}} \sin\left(\tfrac{2}{3}\eta^{\frac{3}{2}} + \pi/4\right)$$

and taking $\eta = K^{\frac{2}{3}} \rho(x)$, we get

$$A_i(-K^{\frac{2}{3}} \rho) \sim \frac{1}{\pi K^{\frac{1}{6}} \rho^{\frac{1}{4}}} \sin\left(\tfrac{2}{3} K \rho^{\frac{3}{2}} + \pi/4\right). \tag{B.10}$$

Comparing the eiconal equation (3.5) with that of (4.5) we see that

$$\tfrac{2}{3} \rho^{\frac{3}{2}}(x) = \sigma(x), \tag{B.11}$$

with the constant of integration chosen to be zero.

Now $B_0(x)$ is given by (4.11)

$$B_0(x) = (h(x)\rho_x)^{-\frac{1}{2}} = \left[\frac{\rho(x)}{h^2(x)(1-q^2)} \right]^{\frac{1}{4}} = \rho(x)^{\frac{1}{4}} (h(x)\sigma_x)^{-\frac{1}{2}} \tag{B.12}$$

where $q = n\pi/h(x)$ and the two last relations follow from (4.5) and (3.5) respectively. Using (3.10) we get:

$$B_0(x) = \text{const.}\, \rho(x)^{\frac{1}{4}} \cdot A_0(x). \tag{B.13}$$

Hence, from (B.10) and (B.11) we get:

$$A_i(-K^{\frac{2}{3}} \rho) B_0(x) \sim \text{const.}\, A_0(x) e^{iK\sigma(x) - i\pi/4}. \tag{B.14}$$

By this we established the equivalence of the first terms in both expansions, far from the turning point.

Since equation (B.14) applies to the general surface $h(x)$, it applies particularly to the canonical problem, which is a special case.

It is impossible to continue the matching higher order terms of the expansion of the exact solution with corresponding terms of the asymptotic solution. To see this, let us focus our attention again on the first terms in both expansions.

The exact solution is $u = H_\nu^{(1)}(Kr) \sin \nu\theta$. For $(Kr - \nu)^3 \gg Kr$ we have

$$H_\nu^1(Kr) \sim \left(\frac{2}{\pi K r} \right)^{\frac{1}{2}} \left[1 - \left(\frac{\nu}{Kr} \right)^2 \right]^{-\frac{1}{4}} \exp \left[iKr \left\{ \left[1 - \left(\frac{\nu}{Kr} \right)^2 \right]^{\frac{1}{2}} - \frac{\nu}{Kr} \cos^{-1} \frac{\nu}{r} - i \frac{\pi}{4} \right\} \right] \tag{B.15}$$

which should be compared to (5.11).

The first terms equivalence in both asymptotic expansions has been done under replacement of r in (B.15) by x. However, expanding r in a Taylor series we have:

$$r = x^2 + Y^2 \cong x \left(1 + \frac{1}{2} \frac{Y^2}{x^2} + \dots \right).$$

Using this approximation in (B.15) we have, by repeated use of Taylor's expansion,

$$\frac{2}{\pi K r}\left[1-\frac{v^2}{(Kr)^2}\right]^{-\frac{1}{4}} \cong \frac{2}{\pi K x}\left[1-\frac{v^2}{(Kx)^2}\right]^{-\frac{1}{4}} \cdot \left(1-\frac{1}{4}\frac{Y^2}{x^2} + \text{higher order terms in } y^2/x^2\right).$$

Using $y = KY$ we get

$$\frac{2}{\pi K r}\left[1-\frac{v^2}{(Kr)^2}\right]^{-\frac{1}{4}} \cong \frac{2}{\pi K x}\left[1-\left(\frac{v}{Kx}\right)^2\right]^{-\frac{1}{4}} \cdot \left[1-\frac{y^2}{4x^2}\left(\frac{1}{K^2}\right)\right].$$

We see that by extracting from (B.15) the Y dependence we get back to the (5.6) relation but now terms dependent on y add to higher order terms as in K. Therefore $A_j(x, y)$ $(j > 1)$ in the asymptotic theory cannot correspond to the j-th term of the exact solution expansion, and one has to add to it these extracts from the first term whose denominator contains K in the j-th power.

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